A simple multibody system on a discrete circle

Chengyang Qian   Yaokun Wu

School of Mathematical Sciences
Shanghai Jiao Tong University

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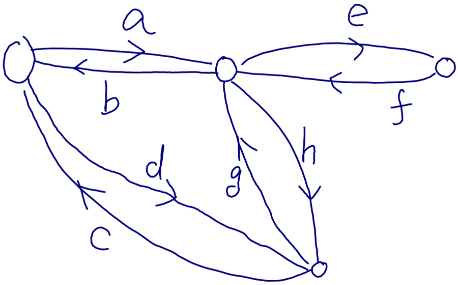
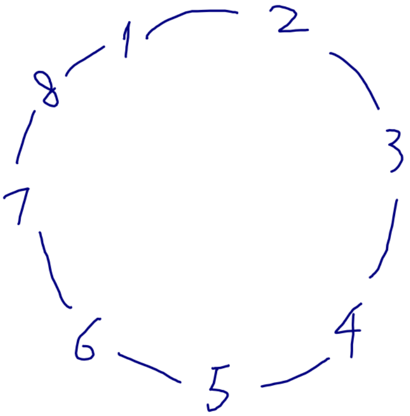
Pointless graphs,

- $ab|cd|ef|gh$,
- $ad|beh|f|cg$.



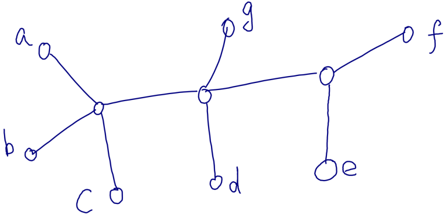
Circular partition, $X = \mathbb{Z}_8$,

- 12|345|67|8,
- 81|2|34|567,
- 812|34|56|7.



Perfect phylogeny,

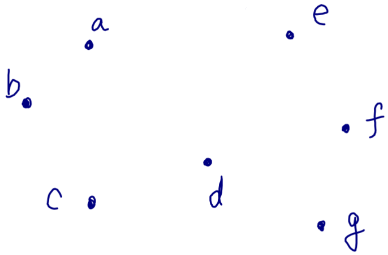
- $abc|def|g$,
- $ad|b|c|g|ef$,
- $ag|b|c|d|ef$.



The subtrees induced by different parts are vertiex disjoint.

Affine partition,

- $abc|de|fg$,
- $abe|df|cg$,
- $ad|bc|efg$.



The convex hulls of different parts are disjoint.

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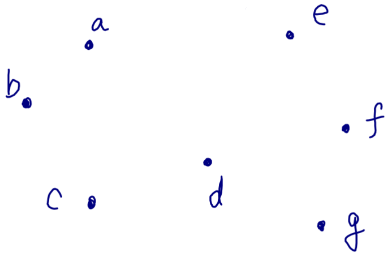

The convex hulls of different parts are disjoint.

For n points in general positions in $\mathbb{R}^k$, the number of different affine splits is

$$\sum_{i=1}^{k} \binom{n-1}{i}.$$

A resolved balanced incomplete-block design,

- $ABC|DEF|GHI$,
- $ADG|BEH|CFI$,
- $AEI|BFG|CDH$,
- $AFH|BDI|CEG$.

[Bai17], Bailey, Relations among partitions, 2017.

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Let $\pi = \pi_1|\pi_2|\ldots|\pi_k$ be a partition of X.

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Definition

For $A \in \binom{X}{t}$, let

$$J_\pi(A) := \begin{cases} 1, & |\pi_i \cap A| \le 1 \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

$$J_\pi^*(A) := \begin{cases} 1, & |\pi_i \cap A| \ge 1 \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Both J_π and J_π^* are $(0,1)$-functions on 2^X, which are also often viewed as functions on $\binom{X}{t}$.
When $k = t$, J_π and J_π^* become equivalent.

Let $\pi = \pi_1|\pi_2|\ldots|\pi_k$ be an ordered partition of X.

Definition

For ordered set $a = (a_1, \ldots, a_t) \in X^{t\downarrow}$, let

$$G_\pi(a) = \begin{cases} 1, & \text{if } k = t, \exists \text{ even permutation } \sigma, a_i \in \pi_{\sigma(i)}, \\ -1, & \text{if } k = t, \exists \text{ odd permutation } \sigma, a_i \in \pi_{\sigma(i)}, \\ 0, & \text{otherwise.} \end{cases}$$

By linear extension, G_π is viewed as a function on $\wedge^k F^X$, the kth exterior power of the linear space F^X over a field F.

Let $\pi = \pi_1|\pi_2|\ldots|\pi_k$ be an ordered partition of X.

Definition

For ordered set $a = (a_1, \ldots, a_t) \in X^{t\downarrow}$, let

$$R_\pi(a) = \begin{cases} 1, & \text{if } k = t, a_i \in \pi_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

By linear extension, R_π is viewed as a function on $(F^X)^k$.

Geometric representation of partition systems

Via the several operators mentioned above, a system of partitions of a set X is represented by a point configuration consisting of $(0, \pm 1)$-vectors in a linear space over $\mathbb{R}$ or other fields.

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- Study the linear span of S.
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  - Construct some natural bases.
  - Find the change-of-basis formula (inversion formula).
- Study the integer linear combinations of S.
  - Calculate the covolume of the lattice.
- Many other ways, say tropical complex.

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We only report some results with simple statements and without heavy notation.

Definition

- Let $S_{n,k}$ be the set of all circular k-partitions of $\mathbb{Z}_n$.
- Let $J_{n,k} = \{J_\pi : \pi \in S_{n,k}\}$.
- Let $G_{n,k} = \{G_\pi : \pi \in S_{n,k}\}$.

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Results

- We determine all the $\binom{n}{k}$ extreme rays of Cone $J_{n,k}$.
- When k is even, we explicitly describe all the $\binom{n}{k}$ facets of Cone $J_{n,k}$.

The elements of Cone $J_{n,2}$ are known as Kalmanson metrics. Our facet description generalizes the classic characterization of Kalmanson metrics.

Kalmanson metric (Chepoi and Fichet, 1998)

A symmetric map d from $X \times X$ to $\mathbb{R}$ is a Kalmanson metric if and only if there exists a circular ordering ξ of X such that

$$d(y,u) + d(z,v) \geq d(y,z) + d(u,v),$$

for all $u, v, y, z \in X$ such that the segments $[y,u]$ and $[z,v]$ are crossing diagonals of the circular ordering of X.

Example

The elements of $f \in \text{Cone}\, J_{n,4}$ are those nonnegative symmetric functions characterized by the inequalities

$$\begin{aligned}
& f(a_1, a_2, a_3, a_4) - f(a_1, a_2, a_3, a_4+1) \\
& - f(a_1, a_2, a_3+1, a_4) + f(a_1, a_2, a_3+1, a_4+1) \\
& - f(a_1, a_2+1, a_3, a_4) + f(a_1, a_2+1, a_3, a_4+1) \\
& + f(a_1, a_2+1, a_3+1, a_4) - f(a_1, a_2+1, a_3+1, a_4+1) \\
& - f(a_1+1, a_2, a_3, a_4) + f(a_1+1, a_2, a_3, a_4+1) \\
& + f(a_1+1, a_2, a_3+1, a_4) - f(a_1+1, a_2, a_3+1, a_4+1) \\
& + f(a_1+1, a_2+1, a_3, a_4) - f(a_1+1, a_2+1, a_3, a_4+1) \\
& - f(a_1+1, a_2+1, a_3+1, a_4) + f(a_1+1, a_2+1, a_3+1, a_4+1) \geq 0
\end{aligned}$$

where $\{a_1, a_2, a_3, a_4\}$ runs through all elements of $\binom{\mathbb{Z}_n}{4}$.

Theorem

The dimension of the linear span of $G_{n,k}$ over a field F is always $\binom{n-1}{k-1}$.

Theorem

The dimension of the linear span of $J_{n,k}$ over a field F is

$$\begin{cases} \binom{n}{k} & \textit{if } k \textit{ is even and } \mathrm{Char}\, F \neq 2, \\ \binom{n-1}{k-1} & \textit{otherwise.} \end{cases}$$

Let $i_1 < i_2 < \cdots < i_k$ be k positive integers no bigger than n. The ordered partition
$i_1, i_1+1, \ldots, i_2-1 | i_2, \ldots, i_3-1 | \ldots | i_k, \ldots, n, 1, \ldots, i_1-1$
is known as a circular k-partition of $\mathbb{Z}_n$ with rotation number one. Let $R_{n,k}$ be the set of all points R_π, where π runs through all circular k-partition of $\mathbb{Z}_n$ with rotation number one.

Theorem

The dimension of the linear span of $R_{n,k}$ over a field F is

$$(k-1)\binom{n}{k} + \binom{n-1}{k-1}.$$

Definition

The even lattice $D^{even}_{X,k}$ is defined to be

$$D^{even}_{X,k} := \left\{ f \in \mathbb{Z}^{\binom{X}{k}} : \sum_{c \in C} f(C \setminus \{c\}) \equiv 0 \pmod 2 \right.$$

$$\left. \text{for all } C \in \binom{X}{k+1} \right\}.$$

The integer linear span of $J_{n,k}$ is denoted as $\Gamma_{n,k}$.

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Theorem

The covolume of $D^{even}_{X,k}$ is always $2^{\binom{n-1}{k}}$. When k is even, it holds $\Gamma_{n,k} = D^{even}_{X,k}$. When k is odd, the covolume of $\Gamma_{n,k}$ in the linear span of $J_{n,k}$ is 1.

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